1.

Complex numbers, a reminder

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A reminder

Complex number operations

As the title suggests, this chapter serves as a reminder of the complex number work you are likely to have covered in earlier units, and that was mentioned in the complex number part of the *Preliminary work* section. However, as we will see later, the chapter also introduces two further ideas, namely the factor theorem and the remainder theorem. Before considering these theorems, first allow the chapter to serve its 'reminder purpose' and work through the following exercise involving complex numbers.

Exercise 1A

1 Write each of the following in the form ai where a is real and $i = \sqrt{-1}$.

a $\sqrt{-64}$ **b** $\sqrt{-8}$ **c** $\sqrt{-10}$ **d** $\sqrt{-63}$

2 For the complex number $z = -5 + 3i$ state

a $\text{Re}(z)$ **b** $\text{Im}(z)$

3 For the complex number $z = 12 - 5i$ state

a $\text{Re}(z)$ **b** $\text{Im}(z)$

4 Use the fact that if $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to determine the *exact* solutions of the following quadratic equations, giving your answers in the form $d + ei$ where d and e are real numbers and $i = \sqrt{-1}$.

a $x^2 - 3x + 3 = 0$ **b** $x^2 + 4x + 7 = 0$

c $3x^2 - x + 1 = 0$ **d** $5x^2 + 8x + 4 = 0$

Simplify each of the following expressions.

5 $(3 + 7i) + (2 - i)$

6 $(1 - 2i) - (3 - 2i)$

7 $12 + 4i - 2 - 5i$

8 $6 - i + 3 + 4i$

9 $(1 + i) + (3 - 2i) + (4 - i)$

10 $2(5 - 2i) + 2(-5 + 3i)$

11 $7(1 - 3i) + 15i$

12 $5 + 3(4 + 2i)$

13 $\text{Re}(5 + 2i) + \text{Re}(-3 + 4i)$

14 $\text{Im}(-1 - 7i) + \text{Im}(3 + 2i)$

15 $(5 - 2i)(2 + 3i)$

16 $(3 + i)(3 + 2i)$

17 $(2 + i)(2 - i)$

18 $(-2 + 7i)(7 - 2i)$

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Express each of the following in the form $a + bi$ where a and b are real numbers.

19 $\frac{2-3i}{1+2i}$

20 $\frac{2-3i}{2+3i}$

21 $\frac{5-2i}{3+4i}$

22 $\frac{i}{2-i}$

23 If $w = 2 + 3i$ and $z = 5 - i$ determine exactly

a $w + z$ **b** $w - z$ **c** $5w - 4z$

d wz **e** z^2 **f** $\frac{w}{z}$

24 If $z = 4 - 7i$ and $\bar{z}$ is the complex conjugate of z determine

a $\bar{z}$ **b** $z + \bar{z}$ **c** $z\bar{z}$ **d** $\frac{z}{\bar{z}}$

25 Given that:

$z = 5 + ai$,
$w = b - 34i$,
a and b are real numbers,
and $z = w$,

determine a and b.

26 If $(a + 5i)(2 - i) = b$ where a and b are real numbers, determine a and b.

27 **a** Use the quadratic formula to prove that if a quadratic equation has any non-real roots then it must have two and they must be conjugates of each other.

b One root of $x^2 + px + q = 0$, for p and q real, is $x = 2 + 3i$. Find p and q.

c One root of $x^2 + dx + e = 0$, for d and e real, is $x = 3 - 2i$. Find d and e.

28 First note the following statements:

- The complex number $a + bi$ can be expressed as the 'ordered pair' (a, b).
- Each of the parts of this question use this ordered pair representation of a complex number.

Simplify the following, giving answers in ordered pair form.

a $(5, 1) + (-3, 2)$

b $(-2, 3) - (1, 3)$

c $(2, 0) \times (2, 1)$

d $(5, -1) \div (-5, 12)$

29 Find all possible real number pairs a, b such that $\frac{14-5i}{a-4i} = 2 + bi$.

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Algebraic fractions

If a fraction has polynomial expressions for both numerator and denominator the fraction may be either 'proper' or 'improper'.

Proper algebraic fractions have the order of the numerator (the top of the fraction) less than the order of the denominator (the bottom of the fraction).

For example: $\dfrac{x}{x^2+3}$, $\dfrac{x^2+3x+4}{x^3+2x+1}$, $\dfrac{x^3+6x^2+2x+1}{x^5-6}$

Improper algebraic fractions have the order of the numerator equal to or greater than the order of the denominator.

For example: $\dfrac{x^2+3x+4}{x+4}$, $\dfrac{x^3+7x^2+5x-1}{x-3}$, $\dfrac{x^2+3x+4}{x^2-2x+1}$, $\dfrac{x^3+6x^2+1}{x^2+3x-2}$

Improper algebraic fractions can, with a bit of 'algebraic juggling', be rearranged to an expression involving a proper fraction.

For example:

$$\begin{aligned}\frac{x^2+3x+4}{x+4} &= \frac{x(x+4)-x+4}{x+4}\\ &= \frac{x(x+4)-(x+4)+8}{x+4}\\ &= x-1+\frac{8}{x+4}\end{aligned}$$

We could say that $(x + 4)$, goes into $x^2 + 3x + 4$, $(x - 1)$ times, with a remainder of 8 still left over the $x + 4$.

$$\begin{aligned}\frac{x^3+7x^2+5x-1}{x-3} &= \frac{x^2(x-3)+10x^2+5x-1}{x-3}\\ &= \frac{x^2(x-3)+10x(x-3)+35x-1}{x-3}\\ &= \frac{x^2(x-3)+10x(x-3)+35(x-3)+104}{x-3}\\ &= x^2+10x+35+\frac{104}{x-3}\end{aligned}$$

We could say that $(x - 3)$, goes into $x^3 + 7x^2 + 5x - 1$, $(x^2 + 10x + 35)$ times, with a remainder of 104 still left over the $x - 3$.

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The remainder theorem

If a polynomial, $f(x)$, is divided by $(x - a)$ until the remainder is a constant (i.e. does not involve x), then this remainder is $f(a)$. This is the **remainder theorem**.

Suppose $f(x) = x^2 + 3x + 4$.

According to the remainder theorem, if we divide $x^2 + 3x + 4$ by $(x + 4)$, i.e. $(x - -4)$, we should find that the remainder is $f(-4)$.

Now if

$$\begin{aligned} f(x) &= x^2 + 3x + 4 \\ f(-4) &= (-4)^2 + 3(-4) + 4 \\ &= 16 - 12 + 4 \\ &= 8 \end{aligned}$$

This is indeed the remainder we obtained when we divided $x^2 + 3x + 4$ by $(x + 4)$ on the previous page.

Suppose $f(x) = x^3 + 7x^2 + 5x - 1$.

According to the remainder theorem, if we divide $x^3 + 7x^2 + 5x - 1$ by $(x - 3)$, we should find that the remainder is $f(3)$.

Now if

$$\begin{aligned} f(x) &= x^3 + 7x^2 + 5x - 1 \\ f(3) &= (3)^3 + 7(3)^2 + 5(3) - 1 \\ &= 27 + 63 + 15 - 1 \\ &= 104 \end{aligned}$$

Again this is the remainder we obtained when we divided $x^3 + 7x^2 + 5x - 1$ by $(x - 3)$ on the previous page.

To prove the remainder theorem we express the polynomial $f(x)$ as the product of $(x - a)$ and some suitable chosen polynomial $g(x)$, with some suitable chosen constant k added,

i.e. $$f(x) = g(x)(x - a) + k.$$

Then, dividing $f(x)$ by $(x - a)$ will leave a remainder of k.

However, if we substitute $x = a$ into $f(x) = g(x)(x - a) + k$

we obtain $f(a) = g(a)(a - a) + k$

i.e. $f(a) = k$

Hence the remainder, k, is equal to $f(a)$, as required.

The factor theorem

From the remainder theorem it follows that for the polynomial $f(x)$, if $f(a) = 0$ then dividing $f(x)$ by $(x - a)$ leaves no remainder, i.e. $(x - a)$ must be a factor of $f(x)$.

Remembering some of the logic symbols encountered in Unit One of the *Mathematics Specialist* course we can write:

If $f(a) = 0$ then $(x - a)$ is a factor of $f(x)$: $f(a) = 0 \Rightarrow (x - a)$ is a factor of $f(x)$.

If $(x - a)$ is a factor of $f(x)$ then $f(a) = 0$: $(x - a)$ is a factor of $f(x) \Rightarrow f(a) = 0$.

$$f(a) = 0 \Leftrightarrow (x - a) \text{ is a factor of } f(x).$$

This is the **factor theorem**.

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EXAMPLE 1

a For $f(x) = x^4 - 6x^3 + 10x^2 + 2x - 15$ determine $f(1), f(-1), f(3)$ and $f(-3)$.

b *Without* using the solve facility of some calculators, solve the following equation.

$$x^4 - 6x^3 + 10x^2 + 2x - 15 = 0$$

Solution

a If

$$\begin{aligned} f(x) &= x^4 - 6x^3 + 10x^2 + 2x - 15 \\ f(1) &= (1)^4 - 6(1)^3 + 10(1)^2 + 2(1) - 15 \\ &= 1 - 6 + 10 + 2 - 15 \\ &= -8 \\ f(-1) &= (-1)^4 - 6(-1)^3 + 10(-1)^2 + 2(-1) - 15 \\ &= 1 + 6 + 10 - 2 - 15 \\ &= 0 \\ f(3) &= (3)^4 - 6(3)^3 + 10(3)^2 + 2(3) - 15 \\ &= 81 - 162 + 90 + 6 - 15 \\ &= 0 \\ f(-3) &= (-3)^4 - 6(-3)^3 + 10(-3)^2 + 2(-3) - 15 \\ &= 81 + 162 + 90 - 6 - 15 \\ &= 312 \end{aligned}$$

Hence, $f(1) = -8, f(-1) = 0, f(3) = 0$ and $f(-3) = 312$.

b With $f(-1) = 0$ and $f(3) = 0$ we know that $(x + 1)$ and $(x - 3)$ are factors of $f(x)$.

$$\begin{aligned} \therefore \quad f(x) &= x^4 - 6x^3 + 10x^2 + 2x - 15 \\ &= (x + 1)(x - 3)(ax^2 + bx + c) \end{aligned}$$

By inspection (consider the x^4 term and the constant term, -15), $a = 1$ and $c = 5$.

$$\begin{aligned} \therefore \quad f(x) &= (x + 1)(x - 3)(x^2 + bx + 5) \\ &= (x^2 - 2x - 3)(x^2 + bx + 5) \end{aligned}$$

Were we to expand the above expression the coefficient of x^2 would be $(5 - 2b - 3)$.

Thus $\quad 5 - 2b - 3 = 10$

Giving $\quad b = -4$

Hence $\quad f(x) = (x + 1)(x - 3)(x^2 - 4x + 5)$

Thus if

$$\begin{aligned} x^4 - 6x^3 + 10x^2 + 2x - 15 &= 0 \\ (x + 1)(x - 3)(x^2 - 4x + 5) &= 0 \end{aligned}$$

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned} \qquad \begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x^2 - 4x + 5 &= 0 \\ x &= \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} \\ &= 2 \pm i \end{aligned}$$

The required solutions are $x = -1, x = 3, x = 2 + i, x = 2 - i$.

Exercise 1B

(In this exercise, when asked to solve an equation, give both real and complex solutions.)

1 Given that $(x - 1)$ is a factor of $2x^3 + x^2 + px + 35$, determine p.

2 Given that $x^3 + 3x^2 - 2x - 16 = (x - a)(bx^2 + cx + d)$, determine a, b, c and d (all $\in \mathbb{R}$).

3 **a** Using the 'algebraic juggling' approach demonstrated on an earlier page, determine the remainder when $x^2 - 7x + 3$ is divided by $(x - 1)$.

b Use the remainder theorem to confirm your answer for part **a**.

4 **a** Using the 'algebraic juggling' approach demonstrated on an earlier page, determine the remainder when $2x^3 + 3x^2 - 4x + 3$ is divided by $(x + 1)$.

b Use the remainder theorem to confirm your answer for part **a**.

5 Find the remainder when $x^2 + 3x - 6$ is divided by $(x - 2)$.

6 Find the remainder when $x^3 - 5x^2 - 8x + 7$ is divided by $(x + 2)$.

7 The function $f(x) = 2x^3 + ax^2 + bx - 2$ has $(2x - 1)$ as a factor but a remainder of -6 is left when $f(x)$ is divided by $(x + 1)$. Find a and b.

8 **a** For $f(x) = x^3 - 3x^2 + 7x - 5$ determine $f(-1)$ and $f(1)$.

b *Without* using the solve facility of some calculators, solve the equation:

$$x^3 - 3x^2 + 7x - 5 = 0$$

c *Without* using the solve facility of some calculators, solve the equation:

$$x^4 - 3x^3 + 7x^2 - 5x = 0$$

9 **a** For $f(x) = x^4 - 5x^3 - x^2 + 11x - 30$ determine $f(-2)$, $f(2)$, $f(-5)$ and $f(5)$.

b *Without* using the solve facility of some calculators, solve the equation:

$$x^4 - 5x^3 - x^2 + 11x - 30 = 0$$

10 **a** For $f(x) = 2x^3 - x^2 + 2x - 1$ determine $f(1)$ and $f(0.5)$.

b *Without* using the solve facility of some calculators, solve the equation:

$$2x^3 - x^2 + 2x - 1 = 0$$

11 *Without* using the solve facility of some calculators, solve the equation:

$$(x^2 + 2x + 2)(x^2 - 2x + 5) = 0$$

12 *Without* using the solve facility of some calculators, solve the equation:

$$2x^3 - 3x^2 + 9x - 8 = 0$$

13 *Without* using the solve facility of some calculators, solve the equation:

$$3x^4 - 3x^3 - 2x^2 + 4x = 0$$

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Miscellaneous exercise one

This miscellaneous exercise may include questions involving the work of this chapter and the ideas mentioned in the Preliminary work section at the beginning of the book.

1 Simplify each of the following.

a $(7 + 3i)(7 - 3i)$

b $(5 + i)(5 - i)$

c $(3 + 2i)(2 - 3i)$

d $(1 - 5i)^2$

e $\dfrac{3 - 2i}{2 + i}$

f $\dfrac{1 + 2i}{3 - 4i}$

2 Given that $z = 3 - 4i$ and $w = -4 + 5i$ determine

a $z + w$

b zw

c $\bar{z}$, the conjugate of z

d z^2

e $\overline{zw}$

f $\bar{z}\,\bar{w}$

g the complex number q such that $\text{Re}(q) = \text{Re}(\bar{w})$
and $\text{Im}(q) = \text{Im}(\bar{z})$.

3 Express $(1 + i)^5$ in the form $a + bi$.

4 Determine $\text{Im}[(1 - 3i)^3]$.

5 Find:

a $\text{Re}(3 - 2i) \times \text{Re}(2 + i)$

b $\text{Re}[(3 - 2i) \times (2 + i)]$

6 Given that $(x - 5)$ is a factor of $x^4 + qx^3 - 14x^2 - 45x - 50$, determine q.

7 Given that $2x^3 - x^2 + 3x + 6 = (x - a)(bx^2 + cx + d)$, determine a, b, c and d (all real).

8 The function $f(x) = x^4 + 3x^3 + px^2 + qx - 30$ has $(x - 3)$ as a factor but a remainder of -48 is left when $f(x)$ is divided by $(x - 1)$. Find p and q.

9 If $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{c} = d\mathbf{i} - 9\mathbf{j}$.
Find

a a vector in the same direction as **a** but twice the magnitude of **a**,

b a vector in the same direction as **b** but equal in magnitude to **a**,

c the possible values of d if **c** has the same magnitude as 3**a**,

d $\mathbf{a}\,.\,\mathbf{b}$

e the angle between **a** and **b**, to the nearest degree.

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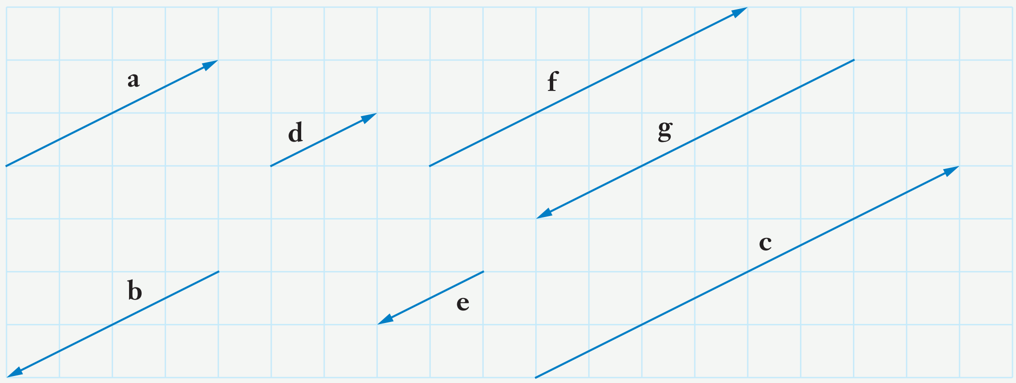
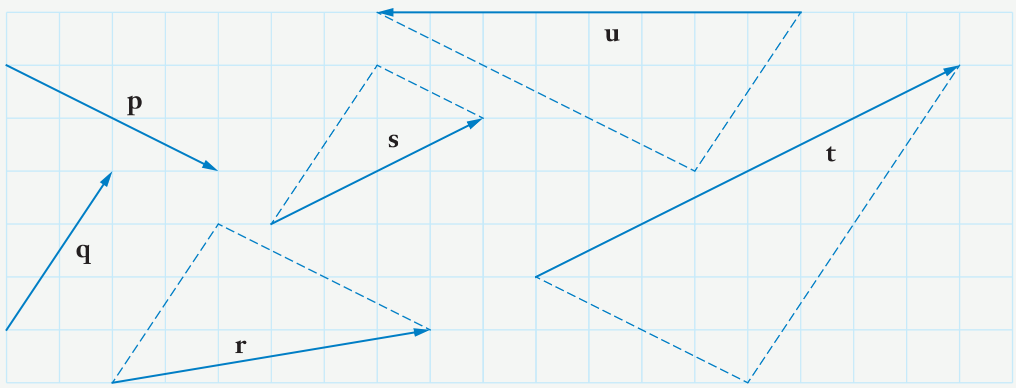
10 With **a** as defined in the diagram below, express each of the vectors **b**, **c**, **d**, **e**, **f** and **g** in terms of **a**.



11 With **p** and **q** as defined in the diagram below, express each of the vectors **r**, **s**, **t** and **u** in terms of **p** and **q**.



12 *Without* using the solve facility of some calculators, solve the equation

$$x^3 + 6x^2 + 4x - 40 = 0.$$

13 For this question the non-zero vectors **p**, **q**, **r** and **s** are such that:

$$\mathbf{p} = a(\mathbf{i} + \mathbf{j}), \quad \mathbf{q} = 2\mathbf{i} - b\mathbf{j}, \quad \mathbf{r} = c\mathbf{i} + d\mathbf{j} \quad \text{and} \quad \mathbf{s} = e\mathbf{i} + f\mathbf{j}.$$

Find the possible values of a, b, c, d, e and f given that all of the following statements are true:

- **p** is perpendicular to **q**.
- $|\mathbf{p}| = |\mathbf{q}|$.
- $\mathbf{q} - 3\mathbf{r} = 23\mathbf{i} - 5\mathbf{j}$.
- **s** is in the same direction as **q** but equal in magnitude to **r**.

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